Notes on Small Aperture Theory

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Abstract *– These informal notes work through the details of the electromagnetic theory of small apertures and their representation by electric and magnetic dipole moments. The treatment is not intended to be comprehensive or fully elaborated, but merely to provide a particular path through the key concepts and steps in the literature to the main results in a consistent formalism.*

Contents

1 Background electromagnetics

1.1 Maxwell's equations and boundary conditions with magnetic sources

The phasor Maxwell equations for time harmonic electromagnetic fields, including both electric and magnetic sources, can be written [Balanis1989]

$$\nabla\cdot\mathbf{D}(\mathbf{r},\omega) = \rho(\mathbf{r},\omega) \tag{1.1}$$
$$\nabla\cdot\mathbf{B}(\mathbf{r},\omega) = \rho_{\mathsf{M}}(\mathbf{r},\omega) \tag{1.2}$$
$$\nabla\times\mathbf{E}(\mathbf{r},\omega) = -\mathrm{j}\omega\mathbf{B}(\mathbf{r},\omega) - \mathbf{J}_{\mathsf{M}}(\mathbf{r},\omega) \tag{1.3}$$
$$\nabla\times\mathbf{H}(\mathbf{r},\omega) = \mathrm{j}\omega\mathbf{D}(\mathbf{r},\omega) + \mathbf{J}(\mathbf{r},\omega), \tag{1.4}$$

where the "engineering" phasor convention

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r},\omega)\mathrm{e}^{+\mathrm{j}\omega t}\right] \tag{1.5}$$

has been used. The symbols and SI units for the electromagnetic quantities used in these notes are summarised in Table 1.

Taking the divergence of the two curl equations, the electric and magnetic currents can be shown to satisfy the continuity equations

$$\nabla\cdot\mathbf{J} + \mathrm{j}\omega\rho = 0 \tag{1.6}$$
$$\nabla\cdot\mathbf{J}_{\mathsf{M}} + \mathrm{j}\omega\rho_{\mathsf{M}} = 0 \tag{1.7}$$

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Symbol	Quantity	Unit
$\mathbf{E}$	Electric field	V m^{-1}
$\mathbf{H}$	Magnetic field	A m^{-1}
$\mathbf{D}$	Electric flux density	C m^{-2}
$\mathbf{B}$	Magnetic flux density	T = Wb m^{-2}
q	Electric charge	C
q_{M}	Magnetic charge	V s = Wb
ρ	Electric charge density	C m^{-3}
ρ_{M}	Magnetic charge density	Wb m^{-3}
$\mathbf{J}$	Electric current density	A m^{-2}
$\mathbf{J}_{\mathsf{M}}$	Magnetic current density	V m^{-2}
I	Electric current	A
I_{M}	Magnetic current	V
$\mathbf{P}$	(Electric) polarisation	C m^{-2}
$\mathbf{M}$	Magnetisation	A m^{-1}
ε	Permittivity	F m^{-1}
μ	Permeability	H m^{-1}
ρ_{s}	Electric surface charge density	C m^{-2}
ρ_{Ms}	Magnetic surface charge density	Wb m^{-2}
$\mathbf{J}_{\mathrm{s}}$	Electric surface current density	A m^{-1}
$\mathbf{J}_{\mathsf{Ms}}$	Magnetic surface current density	V m^{-1}
$\mathbf{p}$, $\mathbf{p}_{\mathsf{M}}$	Electric dipole moment	C m
$\mathbf{m}$, $\mathbf{m}_{\mathsf{M}}$	Magnetic dipole moment	A m^2
$\overline{\boldsymbol{\alpha}}_{\mathrm{e}}$	Electric polarisability tensor	m^3
$\overline{\boldsymbol{\alpha}}_{\mathrm{m}}$	Magnetic polarisability tensot	m^3
$\mathbf{A}$	Magnetic vector potential	Wb m^{-1}
$\mathbf{F}$	Electric vector potential	C m^{-1}

Table 1: Symbols and SI units for electromagnetic quantities.

The electric and magnetic flux densities in homogeneous isotropic media are related to the corresponding fields via the constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon\mathbf{E} + \mathbf{P} \tag{1.8}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu\left(\mathbf{H} + \mathbf{M}\right), \tag{1.9}$$

where ε is the electric permittivity, μ the magnetic permeability and $\mathbf{P}$ and $\mathbf{M}$ are the (electric) polarisation and magnetisation of the medium respectively. The general boundary conditions between two media, medium 1 and medium 2, with the normal vector, $\widehat{\mathbf{n}}$, of the boundary pointing into medium 2 are (Balanis, 1989, Table 1-5)

$$\widehat{\mathbf{n}}\cdot\left(\mathbf{D}_2 - \mathbf{D}_1\right) = \rho_{\mathrm{s}} \tag{1.10}$$

$$\widehat{\mathbf{n}}\cdot\left(\mathbf{B}_2 - \mathbf{B}_1\right) = \rho_{\mathsf{Ms}} \tag{1.11}$$

$$\widehat{\mathbf{n}}\times\left(\mathbf{E}_2 - \mathbf{E}_1\right) = -\mathbf{J}_{\mathsf{Ms}} \tag{1.12}$$

$$\widehat{\mathbf{n}}\times\left(\mathbf{H}_2 - \mathbf{H}_1\right) = \mathbf{J}_{\mathrm{s}}. \tag{1.13}$$

Here the subscript "s" is used to denote surface charges and currents as defined in Table 1. If both media have finite electrical conductivity, $\sigma_1, \sigma_2 \nrightarrow \infty$, and there are no impressed charges on the boundary these reduce to

$$\widehat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 0 \tag{1.14}$$
$$\widehat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{1.15}$$
$$\widehat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0} \tag{1.16}$$
$$\widehat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{0}. \tag{1.17}$$

If medium 1 is a perfect electric conductor (PEC), $\sigma_1 = \infty, \sigma_2 \nrightarrow \infty$, and there are no impressed magnetic sources on the boundary then we have

$$\widehat{\mathbf{n}} \cdot \mathbf{D}_2 = \rho_{\mathrm{s}} \tag{1.18}$$
$$\widehat{\mathbf{n}} \cdot \mathbf{B}_2 = 0 \tag{1.19}$$
$$\widehat{\mathbf{n}} \times \mathbf{E}_2 = \mathbf{0} \tag{1.20}$$
$$\widehat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{J}_{\mathrm{s}}. \tag{1.21}$$

Alternatively, if medium 1 is a perfect magnetic conductor (PMC), $\sigma_{\mathsf{M};1} = \infty, \sigma_{\mathsf{M};2} \nrightarrow \infty$, we get

$$\widehat{\mathbf{n}} \cdot \mathbf{D}_2 = 0 \tag{1.22}$$
$$\widehat{\mathbf{n}} \cdot \mathbf{B}_2 = \rho_{\mathsf{Ms}} \tag{1.23}$$
$$\widehat{\mathbf{n}} \times \mathbf{E}_2 = -\mathbf{J}_{\mathsf{Ms}} \tag{1.24}$$
$$\widehat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{0}. \tag{1.25}$$

1.2 Vector potentials

According to the principle of superposition, the electromagnetic fields in a linear medium due to a collection of electric and magnetic sources can be decomposed into separate contributions due to the electric and magnetic sources alone,

$$\mathbf{E} = \mathbf{E}_{\mathsf{A}} + \mathbf{E}_{\mathsf{F}} \tag{1.26}$$
$$\mathbf{H} = \mathbf{H}_{\mathsf{A}} + \mathbf{H}_{\mathsf{F}}, \tag{1.27}$$

where A denotes the part due to electric source and F the part due to magnetic sources. Helmholtz's theorem shows that each contribution can be written in terms of a scalar and vector potential that determine the irrotational and solenoidal parts of the field (Rothwell and Cloud, 2001, section 5.2). These potentials are the scalar electric potential, φ, and magnetic vector potential, $\mathbf{A}$, for electric sources and the scalar magnetic potential, φ_{M}, and electric vector potential, $\mathbf{F}$, potential for magnetic sources. The fields are given in terms of these potentials by (Balanis, 1989, chapter 6)

$$\mathbf{H}_{\mathsf{A}} = \frac{1}{\mu_0} \nabla \times \mathbf{A} \tag{1.28}$$
$$\mathbf{E}_{\mathsf{A}} = \frac{1}{\mathrm{j}\omega\varepsilon_0} \nabla \times \mathbf{H}_{\mathsf{A}} = -\nabla\varphi - \mathrm{j}\omega\mathbf{A} \tag{1.29}$$
$$\mathbf{E}_{\mathsf{F}} = \frac{-1}{\varepsilon_0} \nabla \times \mathbf{F} \tag{1.30}$$
$$\mathbf{H}_{\mathsf{F}} = \frac{-1}{\mathrm{j}\omega\mu_0} \nabla \times \mathbf{E}_{\mathsf{F}} = -\nabla\varphi_{\mathsf{M}} - \mathrm{j}\omega\mathbf{F}. \tag{1.31}$$

From Maxwell's equations it can then be shown that the potentials satify the wave equations

$$\nabla^2\varphi + \mathrm{j}\omega\nabla \cdot \mathbf{A} = \frac{-\rho}{\varepsilon_0} \tag{1.32}$$
$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu_0\mathbf{J} + \nabla \cdot (\nabla \cdot \mathbf{A} + \mathrm{j}\omega\mu_0\varepsilon_0\varphi) \tag{1.33}$$
$$\nabla^2\varphi_{\mathsf{M}} + \mathrm{j}\omega\nabla \cdot \mathbf{F} = \frac{-\rho_{\mathsf{M}}}{\mu_0} \tag{1.34}$$
$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\varepsilon_0\mathbf{J}_{\mathsf{M}} + \nabla \cdot (\nabla \cdot \mathbf{F} + \mathrm{j}\omega\mu_0\varepsilon_0\varphi_{\mathsf{M}}). \tag{1.35}$$

The scalar and vector potentials are not uniquely determined until a particular gauge is chosen. It is usually convenient to choose the Lorenz gauge,

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \mathrm{j}\omega\mu_0\varepsilon_0\varphi = 0 \tag{1.36}$$
$$\boldsymbol{\nabla}\cdot\mathbf{F} + \mathrm{j}\omega\mu_0\varepsilon_0\varphi_{\mathsf{M}} = 0, \tag{1.37}$$

in which case the fields can be written

$$\mathbf{E}_{\mathsf{A}} = \frac{-j}{\omega\mu_0\varepsilon_0}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) - \mathrm{j}\omega\mathbf{A} = \frac{-\mathrm{j}\omega}{k^2}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) + k^2\mathbf{A}\right] \tag{1.38}$$
$$\mathbf{H}_{\mathsf{F}} = \frac{-j}{\omega\mu_0\varepsilon_0}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}\right) - \mathrm{j}\omega\mathbf{F} = \frac{-\mathrm{j}\omega}{k^2}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}\right) + k^2\mathbf{F}\right] \tag{1.39}$$

and the wave equations for the potentials reduce to

$$\nabla^2\varphi + k^2\varphi = -\rho\varepsilon_0 \tag{1.40}$$
$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu_0\mathbf{J} \tag{1.41}$$
$$\nabla^2\varphi_{\mathsf{M}} + k^2\varphi_{\mathsf{M}} = -\rho_{\mathsf{M}}\mu_0 \tag{1.42}$$
$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\varepsilon_0\mathbf{J}_{\mathsf{M}}. \tag{1.43}$$

In a unbounded space the solutions to these equations are given in terms of the free-space scalar Green's function,

$$G\left(\mathbf{r}\,|\mathbf{r}';\omega\,\right) = \frac{\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}, \tag{1.44}$$

by

$$\varphi(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V' \tag{1.45}$$
$$\mathbf{A}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\iiint_V \frac{\mathbf{J}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V' \tag{1.46}$$
$$\varphi_{\mathsf{M}}(\mathbf{r},\omega) = \frac{1}{4\pi\mu_0}\iiint_V \frac{\rho_M(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V' \tag{1.47}$$
$$\mathbf{F}(\mathbf{r},\omega) = \frac{\varepsilon_0}{4\pi}\iiint_V \frac{\mathbf{J}_{\mathsf{M}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V'. \tag{1.48}$$

The solutions to problems with electric and magnetic sources are related by a duality transformation summarised in Table 2 (Balanis, 1989, Table 7-2).

1.3 Multipole expansion and dipole moments

The vector potentials

$$\mathbf{A}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\iiint_V \frac{\mathbf{J}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V' \tag{1.49}$$
$$\mathbf{F}(\mathbf{r},\omega) = \frac{\varepsilon_0}{4\pi}\iiint_V \frac{\mathbf{J}_{\mathsf{M}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V' \tag{1.50}$$

can be expanded in multipole series. The lowest order terms of these series are the electric and magnetic dipole moment terms given by (Jackson, 1999; Brown, 2007)

$$\mathbf{A}(\mathbf{r},\omega) = \frac{\mathrm{j}\omega\mu_0}{4\pi}\mathbf{p}\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} - \frac{\mathrm{j}k\mu_0}{4\pi}(\widehat{\mathbf{r}}\times\mathbf{m})\left(1+\frac{1}{\mathrm{j}kr}\right)\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \tag{1.51}$$
$$\mathbf{F}(\mathbf{r},\omega) = \frac{\mathrm{j}\omega\mu_0\varepsilon_0}{4\pi}\mathbf{m}_{\mathsf{M}}\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} + \frac{\mathrm{j}k}{4\pi}\left(\widehat{\mathbf{r}}\times\mathbf{p}_{\mathsf{M}}\right)\left(1+\frac{1}{\mathrm{j}kr}\right)\frac{\mathrm{e}^{-\mathrm{j}kr}}{r}, \tag{1.52}$$

Electric Sources	Magnetic sources
$\mathbf{E}$	$\mathbf{H}$
$\mathbf{H}$	$-\mathbf{E}$
ε_0	μ_0
μ_0	ε_0
ρ	ρ_M
$\mathbf{J}$	$\mathbf{J}_\mathsf{M}$
$\mathbf{A}$	$\mathbf{F}$
φ	φ_M
$\mathbf{k}$	$\mathbf{k}$
η_0	$1/\eta_0$
$\mathbf{p}$	$\mu_0\mathbf{m}_\mathsf{M}$
$\mathbf{m}$	$-\mathbf{p}_\mathsf{M}/\varepsilon_0$

Table 2: Duality transformation.

where

$$\mathbf{p}(\omega) \stackrel{\text{def}}{=} \iiint_V \mathbf{r}\rho(\mathbf{r},\omega)\,\mathrm{d}V \tag{1.53}$$

$$\mathbf{m}(\omega) \stackrel{\text{def}}{=} \frac{1}{2}\iiint_V \mathbf{r}\times\mathbf{J}(\mathbf{r},\omega)\,\mathrm{d}V \tag{1.54}$$

are the electric and magnetic dipoles moments for electric sources and

$$\mathbf{p}_\mathsf{M}(\omega) \stackrel{\text{def}}{=} -\frac{\varepsilon_0}{2}\iiint_V \mathbf{r}\times\mathbf{J}_\mathsf{M}(\mathbf{r},\omega)\,\mathrm{d}V \tag{1.55}$$

$$\mathbf{m}_\mathsf{M}(\omega) \stackrel{\text{def}}{=} \frac{1}{\mu_0}\iiint_V \mathbf{r}\rho_\mathsf{M}(\mathbf{r},\omega)\,\mathrm{d}V \tag{1.56}$$

are the electric and magnetic dipoles moments for magnetic sources. The duality transformation for the dipole moments is given in Table 2.

The derivation of these multipole expansions is subtle and complex. It proceeds by expanding the kernel of the vector potential integrals in powers of kr, a so-called “Rayleigh Series”. The lowest order term gives the electric dipole term in $\mathbf{A}$ and the magnetic dipole term in $\mathbf{F}$. The second order terms requires careful analysis using full vector multipole theory (Jackson, 1999, eqn. (9.11)) in order to obtain the correct near-field behaviour and provides the magnetic dipole term in $\mathbf{A}$ and the electric dipole term in $\mathbf{F}$, as well as an electric quadrupole term.

1.4 Alternative representations of dipole moments and currents elements

The electric and magnetic polarisations, $\mathbf{P}$ and $\mathbf{M}$, are dipole moment densities with corresponding electric and magnetic dipole moments

$$\mathbf{p}(\omega) = \iiint_V \mathbf{P}(\mathbf{r},\omega)\,\mathrm{d}V \tag{1.57}$$

$$\mathbf{m}(\omega) = \iiint_V \mathbf{M}(\mathbf{r},\omega)\,\mathrm{d}V. \tag{1.58}$$

For discrete electric charge distributions the dipole moments are summations,

$$\mathbf{p} = \sum_i q_i \mathbf{r}_i \tag{1.59}$$

$$\mathbf{m} = \frac{1}{2} \sum_i q_i \mathbf{r}_i \times \dot{\mathbf{r}}_i, \tag{1.60}$$

while for a collection of discrete dipole moments the corresponding densities are

$$\mathbf{P}(\mathbf{r}, \omega) = \sum_i \mathbf{p}_i(\omega) \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) \tag{1.61}$$

$$\mathbf{M}(\mathbf{r}, \omega) = \sum_i \mathbf{m}_i(\omega) \delta^{(3)}(\mathbf{r} - \mathbf{r}_i). \tag{1.62}$$

The polarisation and magnetisation can be written explicitly in Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{E} = -\mathrm{j}\omega\mu_0 \mathbf{H} - \mathrm{j}\omega\mu_0 \mathbf{M} - \mathbf{J}_{\mathsf{M}} \tag{1.63}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathrm{j}\omega\varepsilon_0 \mathbf{E} + \mathrm{j}\omega \mathbf{P} + \mathbf{J}, \tag{1.64}$$

which shows that the polarisation and magnetisation are equivalent to electric and magnetic current densities given by

$$\mathbf{J}_{\mathsf{M}}(\mathbf{r}, \omega) = \mathrm{j}\omega\mu_0 \mathbf{M}(\mathbf{r}, \omega) \tag{1.65}$$

$$\mathbf{J}(\mathbf{r}, \omega) = \mathrm{j}\omega \mathbf{P}(\mathbf{r}, \omega). \tag{1.66}$$

Current elements (infinitesimal dipoles) are related to dipole moments by

$$\mathbf{J}_{\mathsf{M}} = I_{\mathsf{M}} \, \mathrm{d}\mathbf{l} \, \delta^{(3)}(\mathbf{r} - \mathbf{r}') = \mathrm{j}\omega\mu_0 \mathbf{M} = \mathrm{j}\omega\mu_0 \mathbf{m} \, \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{1.67}$$

$$\mathbf{J} = I \, \mathrm{d}\mathbf{l} \, \delta^{(3)}(\mathbf{r} - \mathbf{r}') = \mathrm{j}\omega \mathbf{P} = \mathrm{j}\omega \mathbf{p} \, \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \tag{1.68}$$

giving

$$I_{\mathsf{M}} \, \mathrm{d}\mathbf{l} = \mathrm{j}\omega\mu_0 \mathbf{m} \tag{1.69}$$

$$I \, \mathrm{d}\mathbf{l} = \mathrm{j}\omega \mathbf{p}. \tag{1.70}$$

A z-directed electric dipole moment can be written

$$\mathbf{p} = q \frac{\mathrm{d}l}{2} \widehat{\mathbf{z}} - (-q) \frac{\mathrm{d}l}{2} \widehat{\mathbf{z}} = q \, \mathrm{d}l \, \widehat{\mathbf{z}} \tag{1.71}$$

and hence

$$I \, \mathrm{d}\mathbf{l} = \mathrm{j}\omega \mathbf{p} = \mathrm{j}\omega q \, \mathrm{d}\mathbf{l} \tag{1.72}$$

so

$$I \, \mathrm{d}l = \mathrm{j}\omega q. \tag{1.73}$$

The time domain Maxwell's equations with explicit polarisation and magnetisation are

$$\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, t) = -\mu_0 \partial_t \mathbf{H}(\mathbf{r}, t) - \mu_0 \partial_t \mathbf{M}(\mathbf{r}, t) - \mathbf{J}_{\mathsf{M}}(\mathbf{r}, t) \tag{1.74}$$

$$\boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r}, t) = \varepsilon_0 \partial_t \mathbf{E}(\mathbf{r}, t) + \partial_t \mathbf{P}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t), \tag{1.75}$$

where for single discrete dipole moments

$$\mathbf{J}_{\mathsf{M}}(\mathbf{r}, t) = \mu_0 \partial_t \mathbf{M}(\mathbf{r}, t) = \mu_0 \partial_t \mathbf{m}(t) \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{1.76}$$

$$\mathbf{J}(\mathbf{r}, t) = \partial_t \mathbf{P}(\mathbf{r}, t) = \partial_t \mathbf{p}(t) \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{1.77}$$

The dipole moments therefore enter the source terms as time derivatives.

1.5 Equivalence of electric current loops and magnetic current elements

The wave equation for the electric field is

$$\nabla\times\nabla\times\mathbf{E}-\omega^2\mu\varepsilon\mathbf{E}=-\mathrm{j}\omega\mu\mathbf{J}-\nabla\times\mathbf{J}_{\mathsf{M}}, \tag{1.78}$$

which shows that *providing the boundary conditions are the same*, an electric current source is equivalent to a magnetic current source if (Nikolova, 2012)

$$\mathrm{j}\omega\mu\mathbf{J}=\nabla\times\mathbf{J}_{\mathsf{M}}. \tag{1.79}$$

Consider the geometry shown in Figure 1. Taking the integral form of this equation on the surface S bounded by the curve C we obtain

$$\mathrm{j}\omega\mu\iint_S\mathbf{J}\cdot\mathrm{d}\mathbf{S}=\oint_C\mathbf{J}_{\mathsf{M}}\cdot\mathrm{d}\mathrm{l}, \tag{1.80}$$

or

$$\mathrm{j}\omega\mu I=J_{\mathsf{M}}\,\mathrm{d}l. \tag{1.81}$$

Hence, a small loop of electric current I circulating around an area A creates the same electromagnetic field as an infinitesimal magnetic dipole moment $I_{\mathsf{M}}\,\mathrm{d}l$ providing that

$$\mathrm{j}\omega\mu IA=I_{\mathsf{M}}\,\mathrm{d}l \tag{1.82}$$

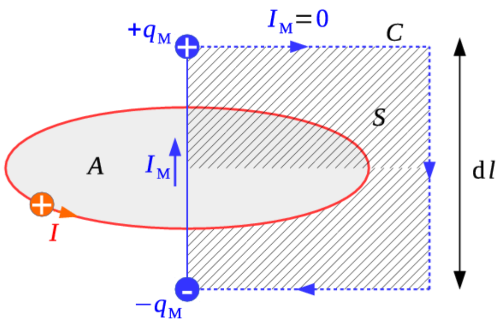

Figure 1: : Equivalence of an electric current loop (red) and a magnetic dipole (blue) (Nikolova, 2012).

1.6 Fields radiated by dipole moments

For electric sources the fields from the dipole moments are determined from the magnetic vector potential using

$$\mathbf{H}(\mathbf{r},\omega)=\frac{1}{\mu}\nabla\times\mathbf{A} \tag{1.83}$$

$$\mathbf{E}(\mathbf{r},\omega)=\frac{1}{\mathrm{j}\omega\varepsilon}\nabla\times\mathbf{H}=-\mathrm{j}\omega\mathbf{A}-\frac{\mathrm{j}}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{A}\right), \tag{1.84}$$

leading to the equations

$$\mathbf{E}^{\mathrm{e}}\left(r,\omega;\mathbf{p}\right)=\frac{k^3}{4\pi\varepsilon_0}\left\{\left[\left(\widehat{\mathbf{r}}\times\mathbf{p}\right)\times\widehat{\mathbf{r}}\right]\frac{1}{kr}+\left[3\widehat{\mathbf{r}}\left(\widehat{\mathbf{r}}\cdot\mathbf{p}\right)-\mathbf{p}\right]\left(\frac{1}{(kr)^3}+\frac{\mathrm{j}}{(kr)^2}\right)\right\}\mathrm{e}^{-\mathrm{j}kr} \tag{1.85}$$

$$\mathbf{H}^{\mathrm{e}}\left(r,\omega;\mathbf{p}\right)=\frac{c_0k^3}{4\pi}\left(\widehat{\mathbf{r}}\times\mathbf{p}\right)\left(1+\frac{1}{\mathrm{j}kr}\right)\frac{\mathrm{e}^{-\mathrm{j}kr}}{kr} \tag{1.86}$$

and

$$\mathbf{E}^{\mathrm{m}}(r;\omega,\mathrm{m}) = \frac{-\eta_0 k^3}{4\pi}(\widehat{\mathbf{r}}\times\mathbf{m})\left(1+\frac{1}{\mathrm{j}kr}\right)\frac{\mathrm{e}^{-\mathrm{j}kr}}{kr} \tag{1.87}$$

$$\mathbf{H}^{\mathrm{m}}(r,\omega;\mathrm{m}) = \frac{k^3}{4\pi}\left\{[(\widehat{\mathbf{r}}\times\mathbf{m})\times\widehat{\mathbf{r}}]\frac{1}{kr}+[3\widehat{\mathbf{r}}(\widehat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}]\left(\frac{1}{(kr)^3}+\frac{\mathrm{j}}{(kr)^2}\right)\right\}\mathrm{e}^{-\mathrm{j}kr}. \tag{1.88}$$

For magnetic sources a similar calculation using the electric vector potential (or more directly using duality) leads to the same equations with $\mathbf{m}$ replaced by $\mathbf{m}_{\mathsf{M}}$ and $\mathbf{p}$ replaced by $\mathbf{p}_{\mathsf{M}}$.

The dipole moment equations are linear. Hence if we have three orthogonal dipole moments, $\mathbf{p}_x = p_x\mathrm{e}^{\mathrm{j}\varphi_x}\widehat{\mathbf{x}}$, $\mathbf{p}_y = p_y\mathrm{e}^{\mathrm{j}\varphi_y}\widehat{\mathbf{y}}$ and $\mathbf{p}_z = p_z\mathrm{e}^{\mathrm{j}\varphi_z}\widehat{\mathbf{z}}$, at the same point then the field from the superposition of dipole moments is the superposition of the individual fields, i.e.

$$\mathbf{E}^{\mathrm{e}}\left(\mathbf{r},\omega;\mathbf{p}_x+\mathbf{p}_y+\mathbf{p}_z\right) = \mathbf{E}^{\mathrm{e}}\left(\mathbf{r},\omega;\mathbf{p}_x\right)+\mathbf{E}^{\mathrm{e}}\left(\mathbf{r},\omega;\mathbf{p}_y\right)+\mathbf{E}^{\mathrm{e}}\left(\mathbf{r},\omega;\mathbf{p}_z\right). \tag{1.89}$$

However, $\mathbf{p} = \mathbf{p}_x+\mathbf{p}_y+\mathbf{p}_z$ *cannot be interpreted* as a single equivalent infinitesimal dipole with moment $\mathbf{p} = p\mathrm{e}^{\mathrm{j}\varphi}\widehat{\mathbf{p}}$ where $\widehat{\mathbf{p}}$ is a real unit vector.

For z-directed electric and magnetic dipoles we can write

$$\mathbf{p} = p_z\widehat{\mathbf{z}} = \frac{I_z\,\mathrm{d}l}{\mathrm{j}\omega}\widehat{\mathbf{z}} \tag{1.90}$$

$$\mathbf{m} = m_z\widehat{\mathbf{z}} = \frac{I_{\mathsf{M}z}\,\mathrm{d}l}{\mathrm{j}\omega\mu_0}\widehat{\mathbf{z}}. \tag{1.91}$$

Using spherical polar coordinates with $\widehat{\mathbf{z}} = \cos\theta\,\widehat{\mathbf{r}}-\sin\theta\,\widehat{\boldsymbol{\theta}}$, we find that

$$(\widehat{\mathbf{r}}\times\mathbf{p})\times\widehat{\mathbf{r}} = p_z(\widehat{\mathbf{r}}\times\widehat{\mathbf{z}})\times\widehat{\mathbf{r}} = -p_z\sin\theta\,\widehat{\boldsymbol{\phi}}\times\widehat{\mathbf{r}} = -p_z\sin\theta\,\widehat{\boldsymbol{\theta}} \tag{1.92}$$

$$3(\widehat{\mathbf{r}}\cdot\mathbf{p})\,\widehat{\mathbf{r}}-\mathbf{p} = p_z[3(\widehat{\mathbf{r}}\cdot\widehat{\mathbf{z}})\,\widehat{\mathbf{r}}-\widehat{\mathbf{z}}] = p_z[3\cos\theta\,\widehat{\mathbf{r}}-\widehat{\mathbf{z}}] = p_z\left[2\cos\theta\,\widehat{\mathbf{r}}+\sin\theta\,\widehat{\boldsymbol{\theta}}\right], \tag{1.93}$$

and similarly for $\mathbf{m}$, giving the field components (Balanis, 1997)

$$E_r^{\mathrm{e}}(\mathbf{r},\omega) = \eta_0\frac{I_z\,\mathrm{d}l\cos\theta}{2\pi r^2}\left(1+\frac{1}{\mathrm{j}kr}\right)\mathrm{e}^{-\mathrm{j}kr} \tag{1.94}$$

$$E_\theta^{\mathrm{e}}(\mathbf{r},\omega) = \mathrm{j}\eta_0\frac{kI_z\,\mathrm{d}l\sin\theta}{4\pi r}\left(1+\frac{1}{\mathrm{j}kr}-\frac{1}{(kr)^2}\right)\mathrm{e}^{-\mathrm{j}kr} \tag{1.95}$$

$$E_\varphi^{\mathrm{e}}(\mathbf{r},\omega) = 0 \tag{1.96}$$

$$H_r^{\mathrm{e}}(\mathbf{r},\omega) = 0 \tag{1.97}$$

$$H_\theta^{\mathrm{e}}(\mathbf{r},\omega) = 0 \tag{1.98}$$

$$H_\varphi^{\mathrm{e}}(\mathbf{r},\omega) = \mathrm{j}\frac{kI_z\,\mathrm{d}l\sin\theta}{4\pi r}\left(1+\frac{1}{\mathrm{j}kr}\right)\mathrm{e}^{-\mathrm{j}kr} \tag{1.99}$$

and

$$E_r^{\mathrm{m}}(\mathbf{r},\omega) = 0 \tag{1.100}$$

$$E_\theta^{\mathrm{m}}(\mathbf{r},\omega) = 0 \tag{1.101}$$

$$E_\varphi^{\mathrm{m}}(\mathbf{r},\omega) = -\mathrm{j}\frac{kI_{\mathsf{M}z}\,\mathrm{d}l\sin\theta}{4\pi r}\left(1+\frac{1}{\mathrm{j}kr}\right)\mathrm{e}^{-\mathrm{j}kr} \tag{1.102}$$

$$H_r^{\mathrm{m}}(\mathbf{r},\omega) = \eta_0^{-1}\frac{I_{\mathsf{M}z}\,\mathrm{d}l\cos\theta}{2\pi r^2}\left(1+\frac{1}{\mathrm{j}kr}\right)\mathrm{e}^{-\mathrm{j}kr} \tag{1.103}$$

$$H_\theta^{\mathrm{m}}(\mathbf{r},\omega) = \mathrm{j}\eta_0^{-1}\frac{kI_{\mathsf{M}z}\,\mathrm{d}l\sin\theta}{4\pi r}\left(1+\frac{1}{\mathrm{j}kr}-\frac{1}{(kr)^2}\right)\mathrm{e}^{-\mathrm{j}kr} \tag{1.104}$$

$$H_\varphi^{\mathrm{m}}(\mathbf{r},\omega) = 0. \tag{1.105}$$

1.7 Far fields and power radiated by dipole moments

In the far field of the dipole moments ($kr \gg 1$) the fields reduce to

$$\mathbf{E}^{\mathrm{e}}(\mathbf{r},\omega) = \frac{k^2}{4\pi\varepsilon_0}\left[(\widehat{\mathbf{r}}\times\mathbf{p})\times\widehat{\mathbf{r}}\right]\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} = \mathrm{j}\eta_0\frac{kI_z\,\mathrm{d}l\sin\theta}{4\pi r}\mathrm{e}^{-\mathrm{j}kr}\widehat{\boldsymbol{\theta}} \tag{1.106}$$

$$\mathbf{H}^{\mathrm{e}}(\mathbf{r},\omega) = \frac{c_0 k^2}{4\pi}\left(\widehat{\mathbf{r}}\times\mathbf{p}\right)\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} = \mathrm{j}\frac{kI_z\,\mathrm{d}l\sin\theta}{4\pi r}\mathrm{e}^{-\mathrm{j}kr}\widehat{\boldsymbol{\phi}} \tag{1.107}$$

and

$$\mathbf{E}^{\mathrm{m}}(\mathbf{r},\omega) = \frac{-\eta_0 k^2}{4\pi}\left(\widehat{\mathbf{r}}\times\mathbf{m}\right)\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} = -\mathrm{j}\frac{kI_{\mathsf{M}z}\,\mathrm{d}l\sin\theta}{4\pi r}\mathrm{e}^{-\mathrm{j}kr}\widehat{\boldsymbol{\phi}} \tag{1.108}$$

$$\mathbf{H}^{\mathrm{m}}(\mathbf{r},\omega) = \frac{k^2}{4\pi}\left[(\widehat{\mathbf{r}}\times\mathbf{m})\times\widehat{\mathbf{r}}\right]\frac{\mathrm{e}^{-\mathrm{j}kr}}{r} = \mathrm{j}\eta_0^{-1}\frac{kI_{\mathsf{M}z}\,\mathrm{d}l\sin\theta}{4\pi r}\mathrm{e}^{-\mathrm{j}kr}\widehat{\boldsymbol{\theta}}, \tag{1.109}$$

where the second form on each line is for a z-directed dipole in spherical polar coordinates. The corresponding Poynting vectors are given by

$$\frac{1}{2}\mathbf{E}^{\mathrm{e}}(\mathbf{r},\omega)\times\mathbf{H}^{\mathrm{e}*}(\mathbf{r},\omega) = c_0^2\eta_0\frac{k^4\left|p_z\right|^2\sin^2\theta}{32\pi^2 r^2}\widehat{\mathbf{r}} = \eta_0\frac{k^2\left|I_z\,\mathrm{d}l\right|^2\sin^2\theta}{32\pi^2 r^2}\widehat{\mathbf{r}} \tag{1.110}$$

$$\frac{1}{2}\mathbf{E}^{\mathrm{m}}(\mathbf{r},\omega)\times\mathbf{H}^{\mathrm{m}*}(\mathbf{r},\omega) = \eta_0\frac{k^4\left|m_z\right|^2\sin^2\theta}{32\pi^2 r^2}\widehat{\mathbf{r}} = \eta_0^{-1}\frac{k^2\left|I_{\mathsf{M}z}\,\mathrm{d}l\right|^2\sin^2\theta}{32\pi^2 r^2}\widehat{\mathbf{r}}. \tag{1.111}$$

Using

$$\oiint_{4\pi}\frac{\sin^2\theta}{r^2}\widehat{\mathbf{r}}\cdot\mathrm{d}S = \int_0^{2\pi}\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 2\pi\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{8\pi}{3} \tag{1.112}$$

we can determine the total power radiated by the dipole moments in the far field as (Jackson, 1999)

$$P^{\mathrm{e}} = c_0^2\eta_0\frac{k^4\left|p\right|^2}{12\pi} = \eta_0\frac{k^2\left|I\,\mathrm{d}l\right|^2}{12\pi} \tag{1.113}$$

$$P^{\mathrm{m}} = \eta_0\frac{k^4\left|m\right|^2}{12\pi} = \eta_0^{-1}\frac{k^2\left|I_{\mathsf{M}}\,\mathrm{d}l\right|^2}{12\pi}. \tag{1.114}$$

2 Coupling through apertures

2.1 Formulation of the aperture coupling problem

Consider an aperture, A, in an infinite perfectly conducting screen, S, in the $x-y$ plane with the aperture centred on $(0,0,0)$ as shown in Figure 2 (Butler et al., 1976, 1978). The background medium is taken to be free-space and a unit vector normal to the screen, $\widehat{\mathbf{n}}$, is defined pointing to the right. In the $z<0$ half-space there are impressed sources $(\mathbf{J}^{\mathrm{i}-},\mathbf{J}_{\mathsf{M}}^{\mathrm{i}-})$ while in the half-space $z>0$ there are impressed sources $(\mathbf{J}^{\mathrm{i}+},\mathbf{J}_{\mathsf{M}}^{\mathrm{i}+})$.

The fields on either side of the screen, $\mathbf{E}^{\pm}(\mathbf{r},\omega)$ and $\mathbf{H}^{\pm}(\mathbf{r},\omega)$, must satisfy Maxwell's equations and the radiation boundary condition for $z\to\pm\infty$. On the surface of the screen the tangential electric field and normal magnetic field are zero. The tangential electric field in the aperture is nonzero and denoted by $\mathbf{E}_{\parallel}^{\mathsf{A}} = \lim_{z\to 0^{\pm}}\mathbf{E}_{\parallel}^{\pm}(\mathbf{r}\in\mathsf{A};\omega)$ and can be regarded as the fundamental unknown in the problem. The tangential magnetic field in the aperture is denoted by $\mathbf{H}_{\parallel}^{\mathsf{A}} = \lim_{z\to 0^{\pm}}\mathbf{H}_{\parallel}^{\pm}(\mathbf{r}\in\mathsf{A};\omega)$. The electric and magnetic fields must also be continuous along any path through the aperture.

The first stage in the solution is to short-circuit the aperture as shown in the left part of Figure 3. The fields in each half space due to the impressed sources with the aperture shorted are called the short circuit fields and denoted by $\mathbf{E}^{\mathrm{sc}\pm}(\mathbf{r},\omega)$ and $\mathbf{H}^{\mathrm{sc}\pm}(\mathbf{r},\omega)$. In particular, at the location of the aperture the normal short-circuit electric fields, $\mathbf{E}_{\perp}^{\mathrm{sc}\pm}(\mathbf{r}\in\mathsf{A};\omega)$, and tangential short-circuit magnetic fields, $\mathbf{H}_{\parallel}^{\mathrm{sc}\pm}(\mathbf{r}\in\mathsf{A};\omega)$, are non-zero and supported by induced electric surface charges and currents on the screen. For the $z<0$ half-space these are given by

$$\mathbf{H}_{\parallel}^{\mathrm{sc}-}(\mathbf{r},\omega) = \widehat{\mathbf{z}}\times\mathbf{H}^{\mathrm{sc}-}(\mathbf{r},\omega) = -\mathbf{J}_{\mathrm{s}}^{-}(\mathbf{r},\omega)\quad \mathbf{r}\in\mathsf{A} \tag{2.1}$$

$$\mathbf{E}_{\perp}^{\mathrm{sc}-}(\mathbf{r},\omega) = \widehat{\mathbf{z}}\cdot\mathbf{E}^{\mathrm{sc}-}(\mathbf{r},\omega) = -\rho_{\mathrm{s}}^{-}(\mathbf{r},\omega)/\epsilon_0\quad \mathbf{r}\in\mathsf{A}. \tag{2.2}$$

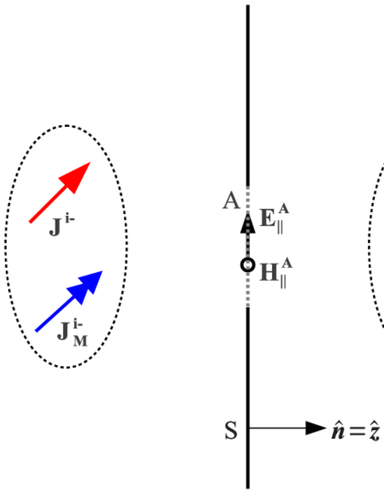

Figure 2: : Geometry of the aperture coupling problem. The aperture, A, is in the x-y plane centred on the origin.

The incident fields, $\mathbf{E}^{\mathrm{i}-}(\mathbf{r},\omega)$ and $\mathbf{H}^{\mathrm{i}-}(\mathbf{r},\omega)$, due to the impressed sources $(\mathbf{J}^{\mathrm{i}-},\mathbf{J}^{\mathrm{i}-}_{\mathsf{M}})$ in $z<0$ are defined as those fields that are present when the sources radiate into unbounded space (Figure 4). At the surface of the PEC screen and in particular at the location of the aperture

$$\mathbf{H}^{\mathrm{sc}-}_{\parallel}(\mathbf{r},\omega)\Big|_{\mathbf{r}\in\mathsf{A}} = 2\,\mathbf{H}^{\mathrm{i}-}_{\parallel}(\mathbf{r},\omega)\Big|_{\mathbf{r}\in\mathsf{A}} \tag{2.3}$$

$$\mathbf{E}^{\mathrm{sc}-}_{\perp}(\mathbf{r},\omega)\big|_{\mathbf{r}\in\mathsf{A}} = 2\,\mathbf{E}^{\mathrm{i}-}_{\perp}(\mathbf{r},\omega)\big|_{\mathbf{r}\in\mathsf{A}}\,. \tag{2.4}$$

Similarly for the incident fields from $z>0$, so together we can write

$$\mathbf{H}^{sc\pm}(\mathbf{r},\omega)\times\widehat{\mathbf{z}}\big|_{\mathbf{r}\in\mathsf{A}} = 2\,\mathbf{H}^{\mathrm{i}\pm}(\mathbf{r},\omega)\times\widehat{\mathbf{z}}\big|_{\mathbf{r}\in\mathsf{A}} \tag{2.5}$$

$$\mathbf{E}^{sc\pm}(\mathbf{r},\omega)\cdot\widehat{\mathbf{z}}\big|_{\mathbf{r}\in\mathsf{A}} = 2\,\mathbf{E}^{\mathrm{i}\pm}(\mathbf{r},\omega)\cdot\widehat{\mathbf{z}}\big|_{\mathbf{r}\in\mathsf{A}}\,. \tag{2.6}$$

From here it is possible to make progress using Babinet's Principle for electromagnetic fields by considering the complementary problem of the fields scattered by a PEC plate occupying the area of the aperture, A, with the screen, S, removed (Bouwkamp, 1954; Tan and McDonald, 2012). In order to restore the fields to the original values the electric charges and currents in the aperture must be "cancelled" by the introduction of opposite currents and charges $\mathbf{J}^{-\prime}_{\mathrm{s}}$ and $\rho^{-\prime}_{\mathrm{s}}$. Again considering the $z<0$ half-space these are given by (Chen and Baum, 1974, eqn. (2.39))

$$\mathbf{J}^{-}_{\mathrm{s}} + \mathbf{J}^{-\prime}_{\mathrm{s}} = 0 \quad \mathbf{r}\in\mathsf{A} \tag{2.7}$$

$$\rho^{-}_{\mathrm{s}} + \rho^{-\prime}_{\mathrm{s}} = 0 \quad \mathbf{r}\in\mathsf{A}. \tag{2.8}$$

However, we proceed instead using the Schelkunoff Equivalence Principle described in (Balanis, 1989, section 7.8) and (Rothwell and Cloud, 2001, section 6.3.4). The Franz form of the vector Huygen's Principle can be written (Rothwell and Cloud, 2001, eqn.(6.41))

$$\mathbf{E}(\mathbf{r},\omega) = \frac{1}{\mathrm{j}\omega\varepsilon_0}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\oiint_{\mathsf{S}}\mathbf{J}_{\mathrm{s}}(\mathbf{r}';\omega)\,G(\mathbf{r}|\mathbf{r}';\omega)\,\mathrm{d}S' - \boldsymbol{\nabla}\times\oiint_{\mathsf{S}}\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)\,G(\mathbf{r}|\mathbf{r}';\omega)\,\mathrm{d}S' \tag{2.9}$$

$$\mathbf{H}(\mathbf{r},\omega) = \frac{1}{\mathrm{j}\omega\mu_0}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\oiint_{\mathsf{S}}\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)\,G(\mathbf{r}|\mathbf{r}';\omega)\,\mathrm{d}S' + \boldsymbol{\nabla}\times\oiint_{\mathsf{S}}\mathbf{J}_{\mathrm{s}}(\mathbf{r}';\omega)\,G(\mathbf{r}|\mathbf{r}';\omega)\,\mathrm{d}S', \tag{2.10}$$

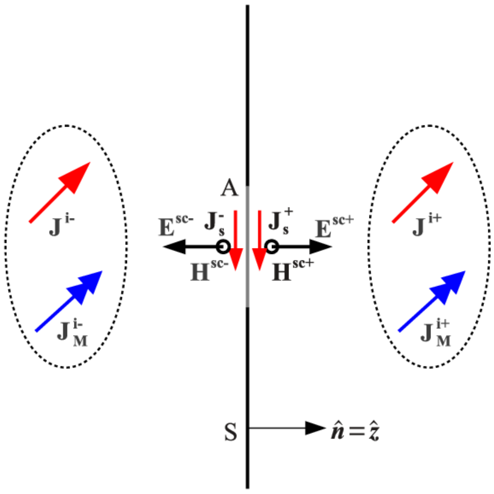

Figure 3: : Definition of the short-circuited fields.

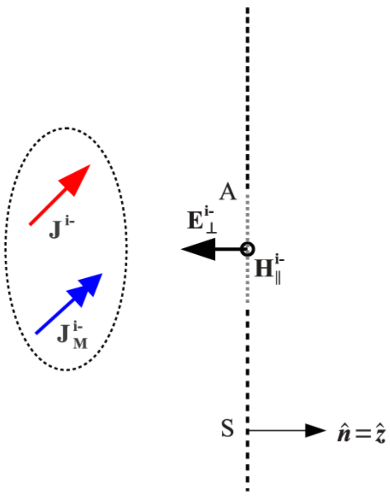

Figure 4: : Definition of the incident fields in the z $<$ 0 half-space.

where $G(\mathbf{r}|\mathbf{r}';\omega)$ is the free space Green's function (1.44),

$$\mathbf{J}_{\mathrm{s}} = \widehat{\mathbf{n}} \times \mathbf{H}|_{\mathbf{r}\in\mathsf{S}} \tag{2.11}$$

$$\mathbf{J}_{\mathsf{Ms}} = -\,\widehat{\mathbf{n}} \times \mathbf{E}|_{\mathbf{r}\in\mathsf{S}}\,, \tag{2.12}$$

and $\widehat{\mathbf{n}}$ is an *inward* normal vector on S. These equations say that the fields in a bounded source free region V (which can be multiply connected) are determined by the tangential fields generated by the sources on the surface S of V. Love's equivalence principle states that the sources outside of V can be replaced by equivalent electric and magnetic current sheets on the surface with the values given by the original tangential fields as above, as described in (Rothwell and Cloud, 2001, section 6.3.3) and (Balanis, 1989, section 7.8). The fields outside V are then zero according to the extinction theorem. An equivalent unbounded problem is thus determined for the fields in V that can be solved using the standard potential techniques. If there are impressed sources within V, then according to the principle of superposition these can be treated separately and their fields added to those generated by the surface currents.

In Love's equivalence principle both electric and magnetic current sheets are required on the bounding surface. According to the uniqueness theorem only one of the tangential fields – electric or magnetic – is required to determine the fields within V (Rothwell and Cloud, 2001, section 4.10.1). We can exploit the fact that the fields outside of V are nullified by the equivalent sources on S to introduce a PEC that fills the excluded space just outside the surface S. Since the field in this region is zero it does not alter the boundary conditions and therefore the fields in V. However the fields in V must now be calculated in the presence of the PEC surface. The PEC surface shorts out the equivalent electric currents leaving only the magnetic currents. This is Schelkunkoff's equivalence principle that we will now apply to the aperture problem.

Again considering the $z < 0$ half-space in the left part of Figure 3. According to Love's equivalence principle the fields in V, which is bounded by $\mathsf{S} \cap \mathsf{A} \cap \mathsf{S}_\infty$ and produced by the sources in the $z > 0$ half-space, can be determined from a surface magnetic current

$$\widehat{\mathbf{z}} \times \mathbf{E}_{\parallel}^{\mathsf{A}}\Big|_{\mathsf{A}} = \mathbf{J}_{\mathsf{Ms}} \tag{2.13}$$

at the location of the aperture, $z = 0^-$, on the shorted PEC plate as shown in Figure 5 (Balanis, 1989, section 7.8). Note that here, in the $z < 0$ half-space the normal $\widehat{\mathbf{z}}$ is pointing out of V so that there is no minus sign in the last equation. The tangential fields on S are zero due to the boundary condition at the PEC surface and the fields on S_∞ vanish due to the radiation boundary condition. Filling the right half-space with PEC we obtain the Schelkunkoff equivalent problem shown in Figure 6. According to the principle of superposition the fields in the left half-space are the sum of the short-circuit fields from the impressed sources in $z < 0$ and the fields from the magnetic current sheet in the aperture.

Finally, using image theory, the infinite sheet can be removed and the images of the magnetic current sheet and impressed sources (in half-space $z < 0$) introduced to maintain the boundary condition of zero tangential field on the $z = 0$ plane as indicated in Figure 7. Since this is now an open homogeneous space the solution, for $z < 0$, can be written using the electric vector potential as

$$\mathbf{H}^-(\mathbf{r},\omega) = \mathbf{H}^{\mathrm{sc}-}(\mathbf{r},\omega) - \frac{\mathrm{j}\omega}{k^2}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}^-\right) + k^2\mathbf{F}^-\right] \quad (z<0), \tag{2.14}$$

where

$$\mathbf{F}^-(\mathbf{r},\omega) = \frac{\varepsilon_0}{4\pi}\iint_{\mathsf{A}} \frac{2\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}S'. \tag{2.15}$$

Note that the magnetic current is doubled due to the presence of its coincident image. The use of the vector potential ensures that the fields satisfy Maxwell's equations and the radiation boundary condition. A similar analysis for the $z > 0$ half-space gives

$$\mathbf{H}^+(\mathbf{r},\omega) = \mathbf{H}^{\mathrm{sc}+}(\mathbf{r},\omega) + \frac{\mathrm{j}\omega}{k^2}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}^+\right) + k^2\mathbf{F}^+\right] \quad (z>0), \tag{2.16}$$

where the sign of the second term has changed since the equivalent magnetic current sheet for the right space is $-\mathbf{J}_{\mathsf{Ms}}$ and we chose to keep the definition

$$\mathbf{F}^+(\mathbf{r},\omega) = \frac{\varepsilon_0}{4\pi}\iint_{\mathsf{A}} \frac{2\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}S'. \tag{2.17}$$

The condition for continuity of the tangential magnetic field through the aperture,

$$\lim_{z\to 0^-}\left[\mathbf{H}^-(\mathbf{r},\omega)\times\widehat{\mathbf{z}}\right] = \lim_{z\to 0^+}\left[\mathbf{H}^+(\mathbf{r},\omega)\times\widehat{\mathbf{z}}\right] \quad \mathbf{r}\in\mathsf{A}, \tag{2.18}$$

then leads to the integro-differential equations for the fields in the aperture,

$$\frac{\mathrm{j}\omega}{k^2}\left[\boldsymbol{\nabla}_{\parallel}\left(\boldsymbol{\nabla}_{\parallel}\cdot\mathbf{F}\right) + k^2\mathbf{F}\right]\times\widehat{\mathbf{z}}\Big|_{\mathbf{r}\in\mathsf{A}} = \frac{1}{2}\left(\mathbf{H}^{\mathrm{sc}-}(\mathbf{r},\omega) - \mathbf{H}^{\mathrm{sc}+}(\mathbf{r},\omega)\right)\times\widehat{\mathbf{z}}\big|_{\mathbf{r}\in\mathsf{A}}, \tag{2.19}$$

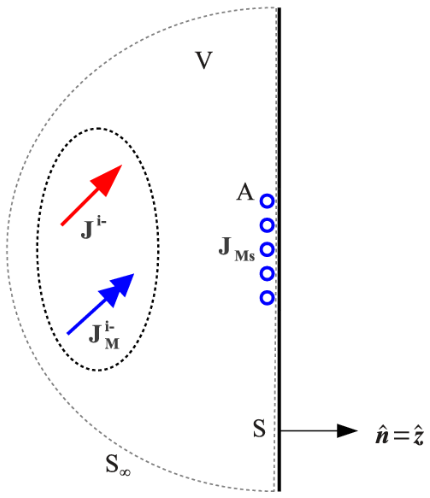

Figure 5: : Equivalent problem for z $<$ 0 half-space with magnetic current sheet replacing aperture.

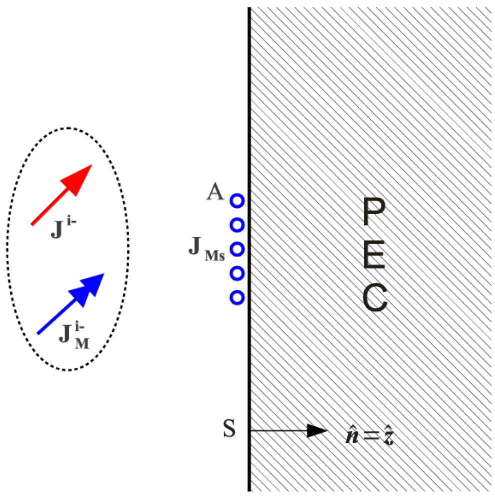

Figure 6: : Schelkunoff equivalent problem with PEC filling the right half-space.

where $\nabla_{\parallel} = \partial_x \widehat{\mathbf{x}} + \partial_y \widehat{\mathbf{y}}$ is the transverse gradient operator. Here

$$\mathbf{F} = \mathbf{F}^{+} + \mathbf{F}^{-}\big|_{z=0}, \tag{2.20}$$

and

$$\mathbf{F}^{\pm}\big|_{z=0} = \lim_{z\to 0^{\pm}} \mathbf{F}^{\pm}(\mathbf{r},\omega) \tag{2.21}$$

are implicitly understood. These integro-differential equations can also be written in terms of the incident fields as:

$$\frac{\mathrm{j}\omega}{k^2}\left[\boldsymbol{\nabla}_{\parallel}\left(\boldsymbol{\nabla}_{\parallel}\cdot\mathbf{F}\right) + k^2\mathbf{F}\right]\times\widehat{\mathbf{z}}\bigg|_{\mathbf{r}\in\mathsf{A}} = \frac{1}{2}\left(\mathbf{H}^{\mathrm{i}-}(\mathbf{r},\omega) - \mathbf{H}^{\mathrm{i}+}(\mathbf{r},\omega)\right)\times\widehat{\mathbf{z}}\big|_{\mathbf{r}\in\mathsf{A}}. \tag{2.22}$$

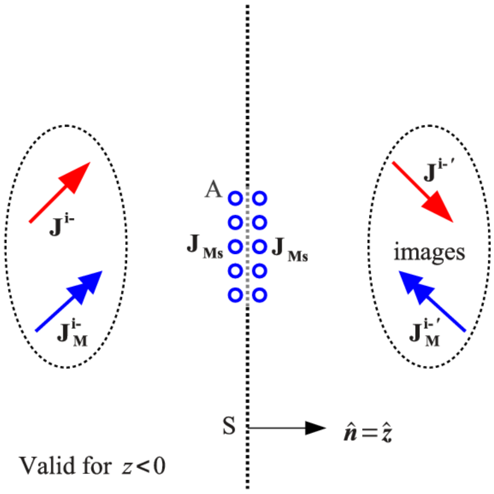

Figure 7: : Equivalent free space problem for the fields in the z $<$ 0 half-space.

Since $\mathbf{J}_{\mathsf{Ms}}$ is tangential to the aperture, both it and $\mathbf{F}$ have no normal component. Hence this vector equation constitutes two coupled integro-differential equations for the two unknown components of $\mathbf{J}_{\mathsf{Ms}}$. Once solved for $\mathbf{J}_{\mathsf{Ms}}$ the magnetic fields can be determined, via $\mathbf{F}^{\pm}$, from

$$\mathbf{H}^{-}(\mathbf{r},\omega) = \mathbf{H}^{\text{sc}-}(\mathbf{r},\omega) - \frac{\text{j}\omega}{k^2}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}^{-}\right) + k^2\mathbf{F}^{-}\right] \quad (z<0) \tag{2.23}$$

$$\mathbf{H}^{+}(\mathbf{r},\omega) = \mathbf{H}^{\text{sc}+}(\mathbf{r},\omega) + \frac{\text{j}\omega}{k^2}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{F}^{+}\right) + k^2\mathbf{F}^{+}\right] \quad (z>0) \tag{2.24}$$

and the electric field from

$$\mathbf{E}^{\pm}(\mathbf{r},\omega) = \mathbf{E}^{\text{sc}\pm}(\mathbf{r},\omega) \pm \frac{1}{\varepsilon_0}\boldsymbol{\nabla}\times\mathbf{F}^{\pm} \quad (z\gtrless 0)\,. \tag{2.25}$$

2.2 Outline solution for small apertures

The potentials due to the magnetic current sheet (with its image) and associated magnetic charge density are given by (Chen and Baum, 1974)

$$\varphi_{\mathsf{M}}(\mathbf{r},\omega) = \frac{1}{4\pi\mu_0}\iint_{\mathsf{A}} \frac{2\rho_{\mathsf{Ms}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\text{e}^{-\text{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\text{d}S' \tag{2.26}$$

$$\mathbf{F}(\mathbf{r},\omega) = \frac{\varepsilon_0}{4\pi}\iint_{\mathsf{A}} \frac{2\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\text{e}^{-\text{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\text{d}S', \tag{2.27}$$

where the magnetic charge density can be found from the equation of continuity

$$\boldsymbol{\nabla}\cdot\mathbf{J}_{\mathsf{Ms}} + \text{j}\omega\rho_{\mathsf{Ms}} = 0. \tag{2.28}$$

The associated electric field and magnetic fields are related to the potentials by

$$\mathbf{E} = \frac{-1}{\varepsilon_0}\boldsymbol{\nabla}\times\mathbf{F} \tag{2.29}$$

$$\mathbf{H} = -\boldsymbol{\nabla}\varphi_M - \text{j}\omega\mathbf{F}. \tag{2.30}$$

For small apertures we assume that the field in the aperture is quasi-static. Hence

$$\mathbf{H}\left(\mathbf{r}\in\mathsf{A};\omega\right) = -\boldsymbol{\nabla}\varphi_{\mathsf{M}} - \text{j}\omega\mathbf{F} \approx -\boldsymbol{\nabla}\varphi_{\mathsf{M}}. \tag{2.31}$$

Assuming the tangential magnetic field, $\mathbf{H}_{\parallel}^{\mathsf{A}}$, and normal electric field, $\mathbf{E}_{\perp}^{\mathsf{A}}$, in the aperture are approximately constant this can be integrated directly to give

$$\varphi_{\mathsf{M}}\left(\mathbf{r}\in\mathsf{A};\omega\right)\approx-\int_{0}^{r}\mathbf{H}_{\parallel}^{\mathsf{A}}\cdot\mathrm{d}\mathbf{r}=-\mathbf{H}_{\parallel}^{\mathsf{A}}\cdot\mathbf{r}. \tag{2.32}$$

Hence the magnetic charge density in the aperture satisfies the integral equation

$$\varphi_{\mathsf{M}}\left(\mathbf{r}\in\mathsf{A};\omega\right)\approx\frac{1}{4\pi\mu_0}\iint_{\mathsf{A}}\frac{2\rho_{\mathsf{Ms}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}S'=-\mathbf{H}_{\parallel}^{\mathsf{A}}\cdot\mathbf{r}, \tag{2.33}$$

where we have approximated $\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\approx 1$ in the quasi-static limit. Note that the electric current density in the shorted aperture is

$$\mathbf{H}_{\parallel}^{\mathsf{A}}=\frac{1}{2}\widehat{\mathbf{n}}\times\mathbf{J}_{\mathrm{s}}. \tag{2.34}$$

In the aperture the electric field is assumed to be normal and also constant, hence from

$$\mathbf{E}_{\perp}^{\mathsf{A}}\left(\mathbf{r}\in\mathsf{A};\omega\right)=\frac{-1}{\varepsilon_0}\boldsymbol{\nabla}\times\mathbf{F} \tag{2.35}$$

we find that

$$\mathbf{F}\left(\mathbf{r}\in\mathsf{A};\omega\right)=\frac{1}{2}\mathbf{E}_{\perp}^{\mathsf{A}}\left(\mathbf{r}\in\mathsf{A};\omega\right)\times\mathbf{r}. \tag{2.36}$$

Note that the electric change density induced on the shorted aperture is

$$\mathbf{E}_{\perp}^{\mathsf{A}}=\frac{\rho_{\mathrm{s}}}{2\varepsilon_0}\widehat{\mathbf{n}}. \tag{2.37}$$

Making the same approximation in the magnetic vector potential we thus obtain the integral equation

$$\mathbf{F}\left(\mathbf{r}\in\mathsf{A};\omega\right)\approx\frac{\varepsilon_0}{4\pi}\iint_{\mathsf{A}}\frac{2\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}S'=\frac{1}{2}\mathbf{E}_{\perp}^{\mathsf{A}}\times\mathbf{r}. \tag{2.38}$$

These integral equations, (2.33) and (2.38), for the magnetic change and current density are equivalent to the integro-differential (2.19) equation derived in the previous section.

This solution for the equivalent magnetic charge and current densities in the aperture is so far a rigorous quasi-static approximation to the small aperture problem. The electric and magnetic fields at all points in space, including in and near the aperture can be determined from them (Chen and Baum, 1974). Bethe also noted that away from the aperture the fields can also be approximated using the equivalent dipole moments of these charges and currents (Bethe, 1944). These are given by

$$\mathbf{p}_{\mathsf{M}}(\mathbf{r},\omega)=\frac{-\varepsilon_0}{2}\iint_{\mathsf{A}}\mathbf{r}'\times\mathbf{J}_{\mathsf{Ms}}\left(\mathbf{r}';\omega\right)\,\mathrm{d}S' \tag{2.39}$$

$$\mathbf{m}_{\mathsf{M}}(\mathbf{r},\omega)=\frac{1}{\mu_0}\iint_{\mathsf{A}}\mathbf{r}'\rho_{\mathsf{Ms}}\left(\mathbf{r}';\omega\right)\,\mathrm{d}S'. \tag{2.40}$$

Since the integral equations are linear the electric and magnetic dipole moments will be linearly related to the tangential magnetic field, $\mathbf{H}_{\parallel}^{\mathsf{A}}$, and normal electric field, $\mathbf{E}_{\perp}^{\mathsf{A}}$, in the aperture respectively. These apertures fields, which are constant in the quasi-static approximation, are half the short circuit fields,

$$2\mathbf{E}_{\perp}^{\mathsf{A}}=\mathbf{E}_{\perp}^{\mathrm{sc}} \tag{2.41}$$

$$2\mathbf{H}_{\parallel}^{\mathsf{A}}=\mathbf{H}_{\parallel}^{\mathrm{sc}}, \tag{2.42}$$

and so the dipole moments can be written as linear functions of the short circuit fields

$$\mathbf{p}_{\mathsf{M}}(\mathbf{r},\omega)=\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\cdot\mathbf{E}^{\mathrm{sc}} \tag{2.43}$$

$$\mathbf{m}_{\mathsf{M}}(\mathbf{r},\omega)=-\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\cdot\mathbf{H}^{\mathrm{sc}}, \tag{2.44}$$

where $\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} = \alpha_{\mathrm{e};zz}\widehat{\mathbf{z}}\widehat{\mathbf{z}}$ and $\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} = \alpha_{\mathrm{m};xx}\widehat{\mathbf{x}}\widehat{\mathbf{x}} + \alpha_{\mathrm{m};yy}\widehat{\mathbf{y}}\widehat{\mathbf{y}}$ are the electric and magnetic polarisability tensors of the aperture. The dipole moments can also be written in terms of current moments (Oates, 1994, p. 70):

$$I\,\mathrm{d}\mathbf{l}(\mathbf{r},\omega) = \mathrm{j}\omega\mathbf{p}_{\mathsf{M}}(\mathbf{r},\omega) = -\mathrm{j}\omega\frac{\varepsilon_0}{2}\iint_{\mathsf{A}} \mathbf{r}'\times\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)\,\mathrm{d}S' = \mathrm{j}\omega\varepsilon_0\alpha_{\mathrm{e}}\mathbf{E}_{\perp}^{\mathrm{sc}} \tag{2.45}$$

$$I_{\mathsf{M}}\,\mathrm{d}\mathbf{l}(\mathbf{r},\omega) = \mathrm{j}\omega\mu_0\mathbf{m}_{\mathsf{M}}(\mathbf{r},\omega) = \mathrm{j}\omega\iint_{\mathsf{A}} \mathbf{r}'\rho_{\mathsf{Ms}}(\mathbf{r}';\omega)\,\mathrm{d}S' = \iint_{\mathsf{A}} \mathbf{J}_{\mathsf{Ms}}(\mathbf{r}';\omega)\,\mathrm{d}S' = -\mathrm{j}\omega\mu_0\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\cdot\mathbf{H}_{\parallel}^{\mathrm{sc}}. \tag{2.46}$$

It must be remembered that the fields from the dipole moments are not valid in the aperture itself, in particular they do not reproduced the correct "knife-edge" field singularities near the edges of the PEC screen (Butler et al., 1976; Chen and Baum, 1974).

2.3 Polarisabilities for some simple aperture shapes

For a circular aperture of radius a the solutions of the integral equations (2.33) and (2.38) are (Bethe, 1944; Chen and Baum, 1974)

$$\rho_{\mathsf{Ms}}(\mathbf{r}\in\mathsf{A};\omega) = \frac{-2\mu_0}{\pi\sqrt{a^2-r^2}}\mathbf{r}\cdot\mathbf{H}_{\parallel}^{\mathsf{A}} \tag{2.47}$$

$$\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}\in\mathsf{A};\omega) = \frac{1}{\pi\sqrt{a^2-r^2}}\mathbf{r}\times\mathbf{E}_{\perp}^{\mathsf{A}} - \frac{\mathrm{j}k}{\pi}\sqrt{a^2-r^2}\mathbf{H}_{\parallel}^{\mathsf{A}}. \tag{2.48}$$

Note that the current contains two terms

$$\mathbf{J}_{\mathsf{Ms}}(\mathbf{r}\in\mathsf{A};\omega) = \mathbf{J}_{\mathsf{Ms}}^{\mathsf{E}}(\mathbf{r}\in\mathsf{A};\omega) + \mathbf{J}_{\mathsf{Ms}}^{\mathrm{H}}(\mathbf{r}\in\mathsf{A};\omega), \tag{2.49}$$

where

$$\boldsymbol{\nabla}\cdot\mathbf{J}_{\mathsf{Ms}}^{\mathrm{H}} + \mathrm{j}\omega\rho_{\mathsf{Ms}} = 0 \tag{2.50}$$

$$\boldsymbol{\nabla}\cdot\mathbf{J}_{\mathsf{Ms}}^{\mathsf{E}} = 0. \tag{2.51}$$

The solenoidal component, $\mathbf{J}_{\mathsf{Ms}}^{\mathsf{E}}$, supports the boundary condition on the normal electric field while the other component, $\mathbf{J}_{\mathsf{Ms}}^{\mathrm{H}}$, supports the boundary condition on the tangential magnetic field.

The corresponding dipole moments of the circular aperture can then be evaluated to give

$$\mathbf{p}_{\mathsf{M}}(\omega) = \varepsilon_0\frac{4a^3}{3}\mathbf{E}_{\perp}^{\mathsf{A}} = \varepsilon_0\frac{2a^3}{3}\mathbf{E}_{\perp}^{\mathrm{sc}} = \varepsilon_0\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\cdot\mathbf{E}^{\mathrm{sc}} \tag{2.52}$$

$$\mathbf{m}_{\mathsf{M}}(\omega) = \frac{-8a^3}{3}\mathbf{H}_{\parallel}^{\mathsf{A}} = \frac{-4a^3}{3}\mathbf{H}_{\parallel}^{\mathrm{sc}} = -\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\cdot\mathbf{H}^{\mathrm{sc}}, \tag{2.53}$$

where the electric and magnetic polarisabilities of the aperture are

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} = \frac{2a^3}{3}\widehat{\mathbf{e}}_{\perp}\widehat{\mathbf{e}}_{\perp} \tag{2.54}$$

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} = \frac{4a^3}{3}\widehat{\mathbf{e}}_{\parallel}\widehat{\mathbf{e}}_{\parallel}. \tag{2.55}$$

The factor of two corresponding to the image of the magnetic current sheet was included in the integrals above, so these are equivalent dipole moments that radiate *in the presence of the PEC screen* to give the fields from the aperture.

The polarisabilities of an elliptical aperture can also be determined analytically (Collin, 1990, p.507, eqn. (70)). For an elliptical aperture with semi-major axis a along the x-direction and semi-minor axis b along the y-direction they are

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} = \frac{\pi}{3}\frac{ab^2}{\mathsf{E}(e)}\widehat{\mathbf{e}}_z \tag{2.56}$$

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} = \frac{\pi}{3}a^3e^2\left(\frac{1}{\mathsf{K}(e)-\mathsf{E}(e)}\widehat{\mathbf{e}}_x\widehat{\mathbf{e}}_x + \frac{1}{(a/b)^2\,\mathsf{E}(e)-\mathsf{K}(e)}\widehat{\mathbf{e}}_y\widehat{\mathbf{e}}_y\right), \tag{2.57}$$

where the eccentricity of the aperture is

$$e = \sqrt{1 - (b/a)^2} \tag{2.58}$$

and the complete elliptic integrals of the first and second kind are

$$\mathsf{K}(e) = \int_0^{\pi/2} \left(1 - e^2 \sin^2\theta\right)^{-1/2} \mathrm{d}\theta \tag{2.59}$$

$$\mathsf{E}(e) = \int_0^{\pi/2} \left(1 - e^2 \sin^2\theta\right)^{1/2} \mathrm{d}\theta. \tag{2.60}$$

For a highly eccentric ellipse with $b \ll a$ $(e \ll 1)$ we find

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} = \frac{\pi}{3} ab^2 \widehat{\mathbf{e}}_z \widehat{\mathbf{e}}_z \tag{2.61}$$

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} = \frac{\pi}{3} \left(\frac{a^3 e^3}{\log_{\mathrm{e}}(4a/b) - 1} \widehat{\mathbf{e}}_x \widehat{\mathbf{e}}_x + ab^2 \widehat{\mathbf{e}}_y \widehat{\mathbf{e}}_y \right). \tag{2.62}$$

Exact analytic expressions for the polarisabilities of square and rectangular apertures do not exist. They can be approximated as circular and elliptical apertures respectively, with the same area and aspect ratio: For a square aperture of side length a this gives

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} = \frac{2a^3}{3\pi^{3/2}} \tag{2.63}$$

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} = \frac{4a^3}{3\pi^{3/2}} \left(\widehat{\mathbf{e}}_x \widehat{\mathbf{e}}_x + \widehat{\mathbf{e}}_y \widehat{\mathbf{e}}_y \right), \tag{2.64}$$

while for a rectangular aperture with side lengths a and b we obtain

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} = \frac{S^{3/2}}{3\sqrt{\pi}} \frac{b/a}{\mathsf{E}(e)} \tag{2.65}$$

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} = \frac{S^{3/2}}{3\sqrt{\pi}} e^2 (a/b)^{3/2} \left(\frac{1}{\mathsf{K}(e) - \mathsf{E}(e)} \widehat{\mathbf{e}}_x \widehat{\mathbf{e}}_x + \frac{1}{(a/b)^2 \mathsf{E}(e) - \mathsf{K}(e)} \widehat{\mathbf{e}}_y \widehat{\mathbf{e}}_y \right), \tag{2.66}$$

where the aperture area is $S = ab$.

Approximate analytical methods to determine the polarisabilities of apertures with arbitrary shapes of low eccentricity are given by (Okon and Harrington, 1981; Fabrikant, 1987a,b). Parametric expressions for aperture polarisabilities determined from measurements can be found in (De Meulenaere and Bladel, 1977; McDonald, 1985, 1987, 1988).

2.4 Equivalent dipole moments for small apertures

Generalising the result of the last section, the effect of a small aperture in a conducting screen can be represented by equivalent electric and magnetic dipole moments on either side of the screen with the aperture short circuited as shown in Figure 8. The dipole moments are related to the short circuit fields by

$$\mathbf{p}^{\pm}(\omega) = \varepsilon_0 \overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} \cdot \left(\mathbf{E}^{\mathrm{sc}\mp}(\mathbf{0};\omega) - \mathbf{E}^{\mathrm{sc}\pm}(\mathbf{0};\omega) \right) \tag{2.67}$$

$$\mathbf{m}^{\pm}(\omega) = -\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} \cdot \left(\mathbf{H}^{\mathrm{sc}\mp}(\mathbf{0};\omega) - \mathbf{H}^{\mathrm{sc}\pm}(\mathbf{0};\omega) \right), \tag{2.68}$$

where the polarisabilities are defined such that $\mathbf{p}^+$ and $\mathbf{m}^+$ are the equivalent dipole moments *in the presence of the screen* for the right half-space $(z > 0)$, i.e. the dipoles located at $(0,0,0^+)$, while the dipoles $\mathbf{p}^-$ and $\mathbf{m}^-$ are located at $(0,0,0^-)$ and radiated into the left half-space $(z < 0)$. This result can be understood physically by considering the behaviour of the electric and magnetic field lines around the aperture as shown in the right part of Figure 8.

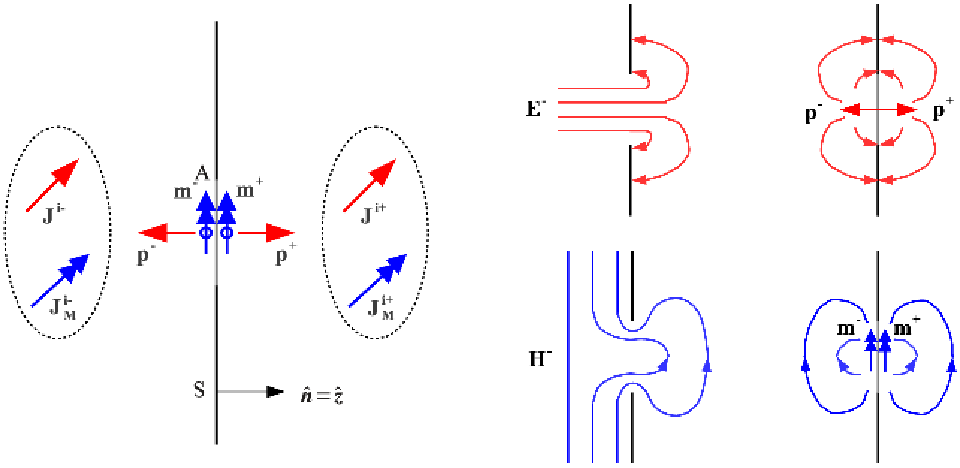

Figure 8: : Equivalent dipole moments to represent the aperture (left) and their interpretation in terms of the electric and magnetic field lines (right).

At points far from the aperture relative to its maximum size the electric and magnetic fields in the two half-spaces are

$$\mathbf{E}^{\pm}(\mathbf{r},\omega) = \mathbf{E}^{\mathrm{sc}\pm}(\mathbf{r},\omega) + \mathbf{E}^{\mathrm{e}}\left(\mathbf{r},\omega;\mathbf{p}^{\pm}\right) + \mathbf{E}^{\mathrm{m}}\left(\mathbf{r},\omega;\mathbf{m}^{\pm}\right) \tag{2.69}$$

$$\mathbf{H}^{\pm}(\mathbf{r},\omega) = \mathbf{H}^{\mathrm{sc}\pm}(\mathbf{r},\omega) + \mathbf{H}^{\mathrm{e}}\left(\mathbf{r},\omega;\mathbf{p}^{\pm}\right) + \mathbf{H}^{\mathrm{m}}\left(\mathbf{r},\omega;\mathbf{m}^{\pm}\right) \tag{2.70}$$

These dipole moments radiate in the presence of a ground plane so their images must be included to give the total field. The short-circuit fields are twice the incident (travelling wave) fields so

$$\mathbf{p}^{\pm}(\omega) = 2\varepsilon_0\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\cdot\left(\mathbf{E}^{\mathrm{i}\mp}(\mathbf{0};\omega) - \mathbf{E}^{\mathrm{i}\pm}(\mathbf{0};\omega)\right) \tag{2.71}$$

$$\mathbf{m}^{\pm}(\omega) = -2\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\cdot\left(\mathbf{H}^{\mathrm{i}\mp}(\mathbf{0};\omega) - \mathbf{H}^{\mathrm{i}\pm}(\mathbf{0};\omega)\right). \tag{2.72}$$

For dipoles radiating in free space without an image the equivalent dipole moments must be doubled to give the same field in the transmitted half-space:

$$\mathbf{p}^{\pm}_{\mathrm{FS}}(\omega) = 4\varepsilon_0\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\cdot\left(\mathbf{E}^{\mathrm{i}\mp}(\mathbf{0};\omega) - \mathbf{E}^{\mathrm{i}\pm}(\mathbf{0};\omega)\right) \tag{2.73}$$

$$\mathbf{m}^{\pm}_{\mathrm{FS}}(\omega) = -4\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\cdot\left(\mathbf{H}^{\mathrm{i}\mp}(\mathbf{0};\omega) - \mathbf{H}^{\mathrm{i}\pm}(\mathbf{0};\omega)\right), \tag{2.74}$$

or with reference to the short-circuited fields

$$\mathbf{p}^{\pm}_{\mathrm{FS}}(\omega) = 2\varepsilon_0\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\cdot\left(\mathbf{E}^{\mathrm{sc}\mp}(\mathbf{0};\omega) - \mathbf{E}^{\mathrm{sc}\pm}(\mathbf{0};\omega)\right) \tag{2.75}$$

$$\mathbf{m}^{\pm}_{\mathrm{FS}}(\omega) = -2\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\cdot\left(\mathbf{H}^{\mathrm{sc}\mp}(\mathbf{0};\omega) - \mathbf{H}^{\mathrm{sc}\pm}(\mathbf{0};\omega)\right), \tag{2.76}$$

Sometime authors include one or both of these factors of two into their definition of the polarisability (Jaggard and Papas, 1977). The definition in (Jackson, 1999, p.424), for example, uses free space dipole moments and short-circuit fields with one factor of two included in the polarisabilities.

2.5 Transmission cross-sections of small apertures

Consider a plane wave incident from $z > 0$ on an aperture located in the $x - y$ plane (centred on the origin) as shown in Figure 9. There are no sources in $z < 0$. The illuminating plane wave propagates with a wave-vector in the direction $\widehat{\mathbf{k}}^{\mathrm{i}}$ defined by

$$\widehat{\mathbf{k}}^{\mathrm{i}} = -\widehat{\mathbf{r}} = \left(-\sin\theta^{\mathrm{i}}\cos\varphi^{\mathrm{i}}, -\sin\theta^{\mathrm{i}}\sin\varphi^{\mathrm{i}}, -\cos\theta^{\mathrm{i}}\right), \tag{2.77}$$

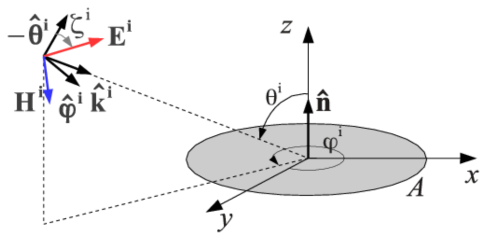

Figure 9: : Plane wave illumination of aperture.

where $\theta^{\rm i}, \varphi^{\rm i}$ are spherical polar angles. The incident electric field vector, $\mathbf{E}^{\rm i}$, is taken to be

$$\mathbf{E}^{\rm i} = E^{\rm i}\left(-\cos\zeta^{\rm i}\widehat{\boldsymbol{\theta}} + \sin\zeta^{\rm i}\widehat{\boldsymbol{\phi}}\right) = E^{\rm i}_\theta\widehat{\boldsymbol{\theta}} + E^{\rm i}_\varphi\widehat{\boldsymbol{\phi}} = E^{\rm i}_\perp\widehat{\mathbf{z}} + \mathbf{E}^{\rm i}_\parallel, \tag{2.78}$$

where $\zeta^{\rm i}$ is the polarisation angle of the electric field relative to the $-\widehat{\boldsymbol{\theta}}$ direction. The magnetic field is thus

$$\mathbf{H}^{\rm i} = \eta_0^{-1}\widehat{\mathbf{k}}^{\rm i}\times\mathbf{E}^{\rm i} = \eta_0^{-1}E^{\rm i}\left(\cos\zeta^{\rm i}\widehat{\boldsymbol{\phi}} + \sin\zeta^{\rm i}\widehat{\boldsymbol{\theta}}\right) = H^{\rm i}_\varphi\widehat{\boldsymbol{\phi}} + H^{\rm i}_\theta\widehat{\boldsymbol{\theta}} = H^{\rm i}_\perp\widehat{\mathbf{z}} + \mathbf{H}^{\rm i}_\parallel. \tag{2.79}$$

Using the transformation matrix from spherical coordinates

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_r \\ A_\theta \\ A_\varphi \end{pmatrix} \tag{2.80}$$

we get the Cartesian form of the incident electric and magnetic fields

$$\mathbf{E}^{\rm i} = -E^i\cos\zeta^i\begin{pmatrix} \cos\theta^{\rm i}\cos\varphi^{\rm i} \\ \cos\theta^{\rm i}\sin\varphi^{\rm i} \\ -\sin\theta^{\rm i} \end{pmatrix} + E^{\rm i}\sin\zeta^{\rm i}\begin{pmatrix} -\sin\varphi^{\rm i} \\ \cos\varphi^{\rm i} \\ 0 \end{pmatrix} \tag{2.81}$$

$$\mathbf{H}^{\rm i} = \eta_0^{-1}E^{\rm i}\cos\zeta^{\rm i}\begin{pmatrix} -\sin\varphi^{\rm i} \\ \cos\varphi^{\rm i} \\ 0 \end{pmatrix} + \eta_0^{-1}E^{\rm i}\sin\zeta^{\rm i}\begin{pmatrix} \cos\theta^{\rm i}\cos\varphi^{\rm i} \\ \cos\theta^{\rm i}\sin\varphi^{\rm i} \\ -\sin\theta^{\rm i} \end{pmatrix}. \tag{2.82}$$

The field components normal and tangential to the plane of the aperture are therefore

$$E^{\rm i}_\perp = E^{\rm i}\cos\zeta^{\rm i}\sin\theta^{\rm i} \tag{2.83}$$

$$\mathbf{E}^{\rm i}_\parallel = -E^{\rm i}\cos\zeta^{\rm i}\cos\theta^{\rm i}\widehat{\boldsymbol{\rho}} + E^{\rm i}\sin\zeta^{\rm i}\widehat{\boldsymbol{\phi}} \tag{2.84}$$

$$H^{\rm i}_\perp = -\eta_0^{-1}E^{\rm i}\sin\zeta^{\rm i}\sin\theta^{\rm i} \tag{2.85}$$

$$\mathbf{H}^{\rm i}_\parallel = \eta_0^{-1}E^{\rm i}\cos\zeta^{\rm i}\widehat{\boldsymbol{\phi}} + \eta + 0^{-1}E^{\rm i}\sin\zeta^{\rm i}\cos\theta^{\rm i}\widehat{\boldsymbol{\rho}}, \tag{2.86}$$

where

$$\widehat{\boldsymbol{\rho}} = \cos\varphi^{\rm i}\widehat{\mathbf{x}} + \sin\varphi^{\rm i}\widehat{\mathbf{y}} \tag{2.87}$$

$$\widehat{\boldsymbol{\phi}} = -\sin\varphi^{\rm i}\widehat{\mathbf{x}} + \cos\varphi^{\rm i}\widehat{\mathbf{y}} \tag{2.88}$$

are plane polar unit vectors in the plane of the aperture. The corresponding Poynting vector is

$$S^{\rm i} = \frac{\left|E^i\right|^2}{2\eta_0}\widehat{\mathbf{k}}^{\rm i} \stackrel{\rm def}{=} S^{\rm i}\widehat{\mathbf{k}}^{\rm i}, \tag{2.89}$$

with magnitude

$$S^{\mathrm{i}} = \frac{1}{2}\eta_0 \left|H^{\mathrm{i}}\right|^2 = \frac{\left|E^{i}\right|^2}{2\eta_0}. \tag{2.90}$$

The incident wave can be considered as a superposition

$$\mathbf{E}^{\mathrm{i}} = \mathbf{E}^{\mathrm{i;TM}} + \mathbf{E}^{\mathrm{i;TE}} \tag{2.91}$$

$$\mathbf{H}^{\mathrm{i}} = \mathbf{H}^{\mathrm{i;TM}} + \mathbf{H}^{\mathrm{i;TE}} \tag{2.92}$$

of TM and TE waves given by

$$\mathbf{E}^{\mathrm{i;TM}} = -E^{\mathrm{i}} \cos\zeta^{\mathrm{i}} \widehat{\boldsymbol{\theta}} \stackrel{\mathrm{def}}{=} E^{\mathrm{i;TM}} \widehat{\boldsymbol{\theta}} \tag{2.93}$$

$$\mathbf{H}^{\mathrm{i;TM}} = \eta_0^{-1} E^{\mathrm{i}} \cos\zeta^{\mathrm{i}} \widehat{\boldsymbol{\phi}} \stackrel{\mathrm{def}}{=} H^{\mathrm{i;TM}} \widehat{\boldsymbol{\phi}} \tag{2.94}$$

and

$$\mathbf{E}^{\mathrm{i;TE}} = E^{\mathrm{i}} \sin\zeta^{\mathrm{i}} \widehat{\boldsymbol{\phi}} \stackrel{\mathrm{def}}{=} E^{\mathrm{i;TE}} \widehat{\boldsymbol{\phi}} \tag{2.95}$$

$$\mathbf{H}^{\mathrm{i;TE}} = \eta_0^{-1} E^{\mathrm{i}} \sin\zeta^{\mathrm{i}} \widehat{\boldsymbol{\theta}} \stackrel{\mathrm{def}}{=} H^{\mathrm{i;TE}} \widehat{\boldsymbol{\theta}}. \tag{2.96}$$

The total power density is given by

$$S^{\mathrm{i}} = S^{\mathrm{i;TM}} + S^{\mathrm{i;TE}}, \tag{2.97}$$

with

$$S^{\mathrm{i;TM}} = \frac{1}{2}\eta_0 \left|\mathbf{H}^{\mathrm{i;TM}}\right|^2 = \frac{\left|\mathbf{E}^{\mathrm{i;TM}}\right|^2}{2\eta_0} \tag{2.98}$$

$$S^{\mathrm{i;TE}} = \frac{1}{2}\eta_0 \left|\mathbf{H}^{\mathrm{i};TE}\right|^2 = \frac{\left|\mathbf{E}^{\mathrm{i};TE}\right|^2}{2\eta_0}. \tag{2.99}$$

Thus we can identify the normal and tangential components with the TM and TE components:

$$E_{\perp}^{\mathrm{i}} = -E^{\mathrm{i;TM}} \sin\theta^{\mathrm{i}} \tag{2.100}$$

$$\mathbf{E}_{\|}^{\mathrm{i}} = E^{\mathrm{i;TM}} \cos\theta^{\mathrm{i}} \widehat{\boldsymbol{\rho}} + E^{\mathrm{i;TE}} \widehat{\boldsymbol{\phi}} \tag{2.101}$$

$$H_{\perp}^{\mathrm{i}} = -H^{\mathrm{i;TE}} \sin\theta^{\mathrm{i}} \tag{2.102}$$

$$\mathbf{H}_{\|}^{\mathrm{i}} = H^{\mathrm{i;TM}} \widehat{\boldsymbol{\phi}} + H^{\mathrm{i;TE}} \cos\theta^{\mathrm{i}} \widehat{\boldsymbol{\rho}}. \tag{2.103}$$

The total power transmitted into the $z < 0$ half-space is

$$P^{\mathrm{t}} = \frac{1}{2} c_0^2 \eta_0 \frac{k^4 \left|\mathbf{p}_{\mathrm{FS}}^{-}\right|^2}{12\pi} + \frac{1}{2}\eta_0 \frac{k^4 \left|\mathbf{m}_{\mathrm{FS}}^{-}\right|^2}{12\pi} \tag{2.104}$$

$$= \frac{\eta_0 k^2}{24\pi} \left\{\omega^2 \left|\mathbf{p}_{\mathrm{FS}}^{-}\right|^2 + k^2 \left|\mathbf{m}_{\mathrm{FS}}^{-}\right|^2\right\} \tag{2.105}$$

$$= \frac{\pi\eta_0}{6\lambda^2} \left\{\omega^2 \left|\mathbf{p}_{\mathrm{FS}}^{-}\right|^2 + k^2 \left|\mathbf{m}_{\mathrm{FS}}^{-}\right|^2\right\} \tag{2.106}$$

where

$$\mathbf{p}_{\mathrm{FS}}^{-}(\omega) = 2\varepsilon_0 \overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} \cdot \mathbf{E}^{\mathrm{sc}+} \tag{2.107}$$

$$\mathbf{m}_{\mathrm{FS}}^{-}(\omega) = -2\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} \cdot \mathbf{H}^{\mathrm{sc}+}. \tag{2.108}$$

Hence we have

$$P^{\mathrm{t}} = \frac{2\pi\eta_0}{3\lambda^2} \left(\omega^2 \varepsilon_0^2 \left|\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}} \cdot \mathbf{E}^{\mathrm{sc}+}\right|^2 + k^2 \left|\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}} \cdot \mathbf{H}^{\mathrm{sc}+}\right|^2\right). \tag{2.109}$$

The short-circuited field components can be evaluated using

$$\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\boldsymbol{\cdot}\mathbf{E}^{\mathrm{sc}+} = \alpha_{\mathrm{e};zz}\widehat{\mathbf{z}}\widehat{\mathbf{z}}\boldsymbol{\cdot}\mathbf{E}^{\mathrm{sc}+} = 2\alpha_{\mathrm{e};zz}(\widehat{\mathbf{z}}\boldsymbol{\cdot}\mathbf{E}^{\mathrm{i}})\widehat{\mathbf{z}} = 2\alpha_{\mathrm{e};zz}E^{\mathrm{i}}\cos\zeta^{\mathrm{i}}\sin\theta^{\mathrm{i}}\widehat{\mathbf{z}} = -2\alpha_{\mathrm{e};zz}E^{\mathrm{i;TM}}\sin\theta^{\mathrm{i}}\widehat{\mathbf{z}} \tag{2.110}$$

and

$$\begin{aligned}\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\boldsymbol{\cdot}\mathbf{H}^{\mathrm{sc}+} &= 2\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\boldsymbol{\cdot}\mathbf{H}^{\mathrm{i}} = 2\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\boldsymbol{\cdot}\mathbf{H}^{\mathrm{i}}_{\parallel} \\ &= 2\alpha_{\mathrm{m};xx}H^{\mathrm{i}}_{x}\widehat{\mathbf{x}} + 2\alpha_{\mathrm{m};yy}H^{\mathrm{i}}_{y}\widehat{\mathbf{y}} = 2\alpha_{\mathrm{m};\varphi\varphi}H^{\mathrm{i;TM}}\widehat{\boldsymbol{\phi}} + 2\alpha_{\mathrm{m};\rho\rho}H^{\mathrm{i;TE}}\cos\theta^{\mathrm{i}}\widehat{\boldsymbol{\rho}}.\end{aligned} \tag{2.111}$$

For the TM case we find (Hill et al., 1994)

$$\mathbf{E}^{\mathrm{i}} = E^{\mathrm{i;TM}}\cos\theta^{\mathrm{i}}\widehat{\boldsymbol{\rho}} - E^{\mathrm{i;TM}}\sin\theta^{\mathrm{i}}\widehat{\mathbf{z}} \tag{2.112}$$
$$\mathbf{H}^{\mathrm{i}} = H^{\mathrm{i};TM}\widehat{\boldsymbol{\phi}} \tag{2.113}$$
$$\mathbf{H}^{\mathrm{sc}-}_{\parallel} = 2H^{\mathrm{i;TM}}\widehat{\boldsymbol{\phi}} \tag{2.114}$$
$$\mathbf{E}^{\mathrm{sc}-}_{\perp} = -2E^{\mathrm{i;TM}}\sin\theta^{\mathrm{i}} \tag{2.115}$$
$$\mathbf{p}^{+}_{\mathrm{FS}}(\omega) = -4\varepsilon_0\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\boldsymbol{\cdot}\widehat{\mathbf{z}}E^{\mathrm{i;TM}}\sin\theta^{\mathrm{i}} \tag{2.116}$$
$$\mathbf{m}^{+}_{\mathrm{FS}}(\omega) = -4\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\boldsymbol{\cdot}\widehat{\boldsymbol{\phi}}H^{\mathrm{i;TM}}. \tag{2.117}$$

The power transmitted into the $z > 0$ half-space is half the total power radiated by the free-space dipole moments,

$$P^{\mathrm{t;TM}} = \frac{1}{2}c_0^2\eta_0\frac{k^4\left|\mathbf{p}^{+}_{\mathrm{FS}}\right|^2}{12\pi} + \frac{1}{2}\eta_0\frac{k^4\left|\mathbf{m}^{+}_{\mathrm{FS}}\right|^2}{12\pi}, \tag{2.118}$$

hence the transmission cross-section for the TM case is

$$\sigma^{\mathrm{t;TM}} = \frac{P^{\mathrm{t;TM}}}{S^{\mathrm{i;TM}}} = \frac{4k^4}{3\pi}\left(\left|\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{e}}\boldsymbol{\cdot}\widehat{\mathbf{z}}\right|^2\sin^2\theta^{\mathrm{i}} + \left|\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\boldsymbol{\cdot}\widehat{\boldsymbol{\phi}}\right|^2\right). \tag{2.119}$$

For the TE case we have

$$\mathbf{H}^{\mathrm{i}} = H^{\mathrm{i;TE}}\cos\theta^{\mathrm{i}}\widehat{\boldsymbol{\rho}} - H^{\mathrm{i;TE}}\sin\theta^{\mathrm{i}}\widehat{\mathbf{z}} \tag{2.120}$$
$$\mathbf{E}^{\mathrm{i}} = E^{\mathrm{i;TE}}\widehat{\boldsymbol{\phi}} \tag{2.121}$$
$$\mathbf{H}^{\mathrm{sc}-}_{\parallel} = H^{\mathrm{i;TE}}\cos\theta^{\mathrm{i}}\widehat{\boldsymbol{\rho}} \tag{2.122}$$
$$\mathbf{E}^{\mathrm{sc}-}_{\perp} = \mathbf{0} \tag{2.123}$$
$$\mathbf{p}^{+}_{\mathrm{FS}}(\omega) = \mathbf{0} \tag{2.124}$$
$$\mathbf{m}^{+}_{\mathrm{FS}}(\omega) = 4\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\boldsymbol{\cdot}\widehat{\boldsymbol{\rho}}H^{\mathrm{i;TE}}\cos\theta^{\mathrm{i}} \tag{2.125}$$
$$P^{\mathrm{t;TE}} = \frac{1}{2}\eta_0\frac{k^4\left|\mathbf{m}^{+}_{\mathrm{FS}}\right|^2}{12\pi} \tag{2.126}$$

giving

$$\sigma^{\mathrm{t;TE}} = \frac{P^{\mathrm{t;TE}}}{S^{\mathrm{i;TE}}} = \frac{4k^4}{3\pi}\left|\overline{\overline{\boldsymbol{\alpha}}}_{\mathrm{m}}\boldsymbol{\cdot}\widehat{\boldsymbol{\rho}}\right|^2\cos^2\theta^{\mathrm{i}}. \tag{2.127}$$

For a circular aperture the cross-sections reduce to (Hill et al., 1994, eqn. (26)-(27))

$$\sigma^{\mathrm{t;TM}} = \frac{64k^4a^6}{27\pi}\left(\frac{1}{4}\sin^2\theta^{\mathrm{i}} + 1\right) \tag{2.128}$$
$$\sigma^{\mathrm{t;TE}} = \frac{64k^4a^6}{27\pi}\cos^2\theta^{\mathrm{i}}. \tag{2.129}$$

References

C. A. Balanis. *Advanced Engineering Electromagnetics*. John Wiley, New York, 1989.

C. A. Balanis. *Antenna Theory*. John Wiley, New York, 2nd edition, 1997.

H. A. Bethe. Theory of diffraction by small holes. *Physical Review*, 66:163–182, Oct 1944. doi:10.1103/PhysRev.66.163.

C. J. Bouwkamp. Diffraction theory. *Reports on Progress in Physics*, 17(1):35, 1954. doi:10.1088/0034-4885/17/1/302.

R. G. Brown. Vector calculus: Integration by parts. URL:, Dec. 2007, url: `http://www.phy.duke.edu/~rgb/Class/phy319/phy319/node14.html`.

C. Butler, Y. Rahmat-Samii, and R. Mittra. A review of electromagnetic penetration through apertures in conducting surfaces for emp applications. Interaction Note 308, SUMMA Foundation, Aug. 1976, url: `http://ece-research.unm.edu/summa/notes/In/0308.pdf`.

C. Butler, Y. Rahmat-Samii, and R. Mittra. Electromagnetic penetration through apertures in conducting surfaces. *IEEE Transactions on Antennas and Propagation*, 26(1):82–93, Jan 1978. doi:10.1109/TAP.1978.1141788.

K. C. Chen and C. E. Baum. On EMP excitations of cavities with small openings. Interaction Note 170, SUMMA Foundation, Jan. 1974, url: `http://ece-research.unm.edu/summa/notes/In/0170.pdf`.

R. E. Collin. *Field Theory of Guided Waves*. IEEE/OUP Series on Electromagnetic Wave Theory. IEEE Press, Piscataway, NJ, 1990.

F. De Meulenaere and J. V. Bladel. Polarizability of some small apertures. *IEEE Transactions on Antennas and Propagation*, 25(2):198–205, Mar 1977. doi:10.1109/TAP.1977.1141568.

V. I. Fabrikant. Magnetic polarisability of small apertures: analytical approach. *Journal of Physics A: Mathematical and General*, 20(2):323, 1987a. doi:10.1088/0305-4470/20/2/018.

V. I. Fabrikant. Electrical polarizability of small apertures: analytical approach. *International Journal of Electronics*, 62(4):533–545, 1987b. doi:10.1080/00207218708921004.

D. A. Hill, M. T. Ma, A. R. Ondrejka, B. F. Riddle, M. L. Crawford, and R. T. Johnk. Aperture excitation of electrically large, lossy cavities. *IEEE Transactions on Electromagnetic Compatibility*, 36 (3):169–178, Aug 1994. doi:10.1109/15.305461.

J. D. Jackson. *Classical Electrodynamics*. Jon Wiley, 3rd edition, 1999.

D. L. Jaggard and C. H. Papas. On the application of symmetrization to the transmission of electromagnetic waves through small convex apertures of arbitrary shape. Interaction Note 324, SUMMA Foundation, Oct. 1977, url: `http://ece-research.unm.edu/summa/notes/In/0324.pdf`.

N. A. McDonald. Polynomial approximations for the electric polarizabilities of some small apertures. *IEEE Transactions on Microwave Theory and Techniques*, 33(11):1146–1149, Nov 1985. doi:10.1109/TMTT.1985.1133186.

N. A. McDonald. Polynomial approximations for the transverse magnetic polarizabilities of some small apertures. *IEEE Transactions on Microwave Theory and Techniques*, 35(1):20–23, Jan 1987. doi:10.1109/TMTT.1987.1133589.

N. A. McDonald. Simple approximations for the longitudinal magnetic polarizabilities of some small apertures. *IEEE Transactions on Microwave Theory and Techniques*, 36(7):1141–1144, Jul 1988. doi:10.1109/22.3648.

N. K. Nikolova. Radiation from infinitesimal (elementary) sources. Lecture notes, ECE753 Modern Antennas in Wireless Telecommunications, Jan. 2012, url: `http://www.ece.mcmaster.ca/faculty/nikolova/antenna_dload/current_lectures/L03_RadIS.pdf`.

J. H. Oates. *Propagation and scattering of electromagnetic waves in complex environments*. PhD thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 1994, url: https://dspace.mit.edu/bitstream/handle/1721.1/34062/31262929-MIT.pdf.

E. E. Okon and R. F. Harrington. The polarizabilities of electrically small apertures of arbitrary shape. *IEEE Transactions on Electromagnetic Compatibility*, 23(4):359–366, Nov 1981. doi:10.1109/TEMC.1981.303968.

E. J. Rothwell and M. J. Cloud. *Electromagnetics*. Electrical Engineering Text Book Series. CRC Press, 2001.

Z. M. Tan and K. T. McDonald. Babinet's principle for electromagnetic fields. Joseph Henry Laboratories, Princeton University, Princeton, NJ, Jan. 2012, url: http://www.physics.princeton.edu/~mcdonald/examples/babinet.pdf.